How Many Rolls Will It Take? Three Probability Activities for Your Senior High Mathematics Classroom

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[Note: Activity 1 and Activity 2 are in the 2008–2009 Special Edition of the ComMuniCator, Probability and Statistics Across the Grades, K–12.]

Activity 3. A die is rolled repeatedly until some number appears twice *in succession.* On the average, how many rolls will this take?

A. Find the average experimentally.

Again we begin by displaying the question on the overhead projector and recording student guesses. Students will work in pairs, one as die roller and the other as data recorder. As was done with Activities 1 and 2, five to ten trials per student pair should be sufficient. Because the students need to get matching numbers on consecutive rolls, the Student Record (*Figure 5*) is quite simple. There is no need to keep a record of what number appears on each roll, only to note and record how many rolls it took until some number appeared twice in succession. Have an overhead Class Record transparency ready for students to record their results on. The Class Record (*Figure 6*) is from two classes combined.

Trial Number	Number of Rolls until Same Number Appeared Twice in Succession	Tally
1	| | | | | | | | | | | |	12
2	| | | |	4
3	| |	2
4	| |	2
5	| | | | | | |	7

Figure 5: Student Record (sample)

Number of Rolls Required	Frequency (tallied)	Number of Rolls Required	Frequency (tallied)	Number of Rolls Required	Frequency (tallied)
2	36	11	2	20	1
3	22	12	1	21	
4	18	13	4	22	
5	17	14	4	23	
6	12	15	3	24	1
7	13	16	1	25	1
8	9	17	2	26	
9	3	18	3	27	
10	6	19	1	28	

Figure 6: Class Record

The combined classes' average number of rolls will be the total number of rolls divided by the total number of trials. For the data in *Figure 6*, the total number of trials is 160 (the frequency sum), and the total number of rolls is the sum of the products: frequency x number of rolls required = 36 x 2 + 22 x 3 + 18 x 4 + … + 1 x 25 = 996. The experimental class average equals $996 \div 160 \approx 6.23$ rolls. As was done in Activity 2, we could have gotten this same average by weighting each "number of rolls required" as follows:

$$\text{Experimental class average} = \frac{36}{160} \times 2 + \frac{22}{160} \times 3 + \frac{18}{160} \times 4 + \ldots + \frac{1}{160} \times 25.$$

This calculation results in the same 6.23 average number of rolls required until some number appears twice in succession. Again it is time to acknowledge/reward the student whose guess came closest to the class average.

B. Find the average by calculator simulation.

The following program is written for a TI-83. You might choose the same number of trials that was done experimentally by your class and then display the program run on an overhead calculator to hold students' attention. When I was writing this article, I ran the program for 160 trials twice and got averages of approximately 7.49 rolls and 6.83 rolls.

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Program:Die3
:ClrHome                      :Lbl 1                :S + C→S
:Disp "HOW MANY TRIALS?"      :R→P                  :End
:Prompt N                     :randInt(1,6)→R       :Disp "TOTAL TRIALS", N
:0→S                          :C + 1→C              :Disp "TOTAL ROLLS", S
:For(I,1,N)                   :If R ≠ P             :Disp "AVERAGE", S/N
:0→C                          :Goto 1
:0→R                          :Disp C
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C. Find the average mathematically.

We begin by finding the probabilities for getting a match on consecutive roll numbers for the first time, for each possible roll number, beginning with roll 2. That probability is 1/6, because the second roll outcome has 1 chance in 6 of matching the first roll outcome. We will use the product rule for independent events to determine the probabilities for subsequent roll numbers. For a consecutive match first occurring on roll 3, we use the probability that second roll did not match first roll, probability 5/6, and that third roll matches second roll, probability 1/6. Therefore the probability that the first consecutive match occurs on roll 3 is 5/6 x 1/6 = 5/36. For a consecutive match first occurring on roll 4, we use the probability that second roll did not match first roll, probability 5/6, that third roll did not match second roll, probability 5/6, and that fourth roll matches third roll, probability 1/6. Therefore the probability that the first consecutive match occurs on roll 4 is 5/6 x 5/6 x 1/6 = $5^2/6^3$. Similarly, the probability that the first consecutive match occurs on roll 5 is 5/6 x 5/6 x 5/6 x 1/6 = $5^3/6^4$. In general, the probability that the first consecutive match occurs on roll n is $5^{n-2}/6^{n-1}$. As we did in Activity 2 part C, we will use these probabilities as weights with their corresponding roll numbers in order to determine the theoretical average number of rolls for the problem. The average number of rolls needed will be $(1/6)2 + (5/6^2)3 + (5^2/6^3)4 + (5^3/6^4)5 + \ldots + (5^{n-2}/6^{n-1})n + \ldots$ This sum has an infinite number of terms, because, in theory, a match on consecutive rolls might never occur. If we rewrite the first term of this infinite series as $(5^0/6^1)2$, then we can rewrite the series in condensed form: $\sum_{n=2}^{\infty} [(5^{n-2}/6^{n-1})n]$. This is a good place for students to use the summation function on their graphing calculators to evaluate the sum for a few large values of n. Using a TI-83, the sum of the first 30 terms is 6.818010486. For $n = 50$, the sum is 6.99261574 and for $n = 100$, the sum is 6.999998464. With these, or even a few larger values for n, we hope that most students will see that the series converges to 7.

What about a more rigorous approach? The series seems to resemble an infinite geometric series, and with a bit of rewriting and some basic calculus, there is a way to prove that the sum is indeed 7. We start by rewriting the series a couple of times:

$$\sum_{n=2}^{\infty} [(5^{n-2}/6^{n-1})n] = \sum_{n=2}^{\infty} [(1/5)(5^{n-1})/6^{n-1})n] = (1/5)\sum_{n=2}^{\infty} [n(5/6)^{n-1}].$$ Except for the factor of n, this

looks very much like an infinite geometric series with common ratio 5/6. We know that an

infinite geometric series will converge absolutely when the absolute value of the common ratio is less than 1. Specifically, the infinite geometric series, $x^0 + x^1 + x^2 + x^3 + ... + x^n + ... = \dfrac{1}{1-x}$ will converge, if $|x| < 1$. If we take the derivative of each side of this equation with respect to x, we will get equal results. The derivative of the left hand side sum is the sum of the derivatives of the individual terms. Applying the power rule for differentiation to each term, that derivative is $0 + 1(x^0) + 2(x^1) + 3(x^2) + ... + n(x^{n-1}) + ...$, where $|x| < 1$. This result can be written as $\displaystyle\sum_{n=1}^{\infty}[n(x^{n-1})]$, where $|x| < 1$. The derivative of the right hand side can be found by applying the quotient rule for differentiation, $D_x\left(\dfrac{u}{v}\right) = \dfrac{v(D_x u) - u(D_x v)}{v^2}$. For our quotient

$$D_x\left(\frac{1}{1-x}\right) = \frac{(1-x)(D_x 1) - (1)[D_x(1-x)]}{(1-x)^2} = \frac{(1-x)(0) - (1)(0-1)}{(1-x)^2} = \frac{1}{(1-x)^2},$$ where $|x| < 1$. Therefore

$$\sum_{n=1}^{\infty}[n(x^{n-1})] = \frac{1}{(1-x)^2}$$, where $|x| < 1$. Substituting $5/6$ for x gives us

$$\sum_{n=1}^{\infty}[n(5/6)^{n-1}] = \frac{1}{[1-(5/6)]^2} = \frac{1}{(1/6)^2} = \frac{1}{1/36} = 36.$$ Because

$$\sum_{n=1}^{\infty}[n(5/6)^{n-1}] = 1(5/6)^0 + \sum_{n=2}^{\infty}[n(5/6)^{n-1}] = 1 + \sum_{n=2}^{\infty}[n(5/6)^{n-1}] = 36,$$ we have $\displaystyle\sum_{n=2}^{\infty}[n(5/6)^{n-1}] = 35$. Now

we multiply both sides of this last equation by $1/5$ to get $(1/5)\displaystyle\sum_{n=2}^{\infty}[n(5/6)^{n-1}] = 7$. Because the left side of this equation was shown to be equivalent to our original series, the statement is proved. Thus on the average, exactly 7 rolls of a die are needed until some number appears twice *in succession*.

References
Isaac, Richard. *The Pleasures of Probability*. New York: Springer, 1995.